Stability of the Triangular Points in the Elliptic Restricted Problem of Three Bodies

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A perturbation scheme is used to study the stability of infinitesimal motions about the triangular points in the elliptic restricted problem of three bodies. Fourth-order analytical expressions for the transition curves that separate stable from unstable orbits in the μ-e plane are given. These power series are recast into rational fractions to extend their validity to larger values of eccentricity, e.

1. Introduction

DANBY[3] studied the linear stability of the triangular points numerically using Floquet theory. He presented transition curves that separate the stable from the unstable orbits in the μ-e plane (μ is the ratio of the smaller primary to the sum of the masses of the two primaries, and e is the eccentricity of the primaries' orbit). These curves intersect the μ axis at μ_a and μ_b, where $\mu_a = 0.03852$ is the limiting value of μ for stable orbits in the circular case, and $\mu_b = 0.02859$ is the value of μ such that one of the periods of motion about the triangular points is exactly twice the period of the orbit of the primaries in the circular case.

Bennet[2] obtained a first-order analytical expression for the transition curves near μ_b using an analytical technique for determination of characteristic exponents. Alfriend and Rand[1] obtained second-order analytical expressions for the transition curves at μ_a and μ_b using the method of multiple scales.[5–7]

In this paper, we use a perturbation technique to determine fourth-order analytical expressions for the transition curves. We use this technique rather than the method of multiple scales because we are interested in determining the transition curves only. The amount of algebra involved is considerably less than that required if we use the method of multiple scales because the latter provides the solution in the whole μ-e plane. The expansions obtained here are recast into rational fractions[8] to extend their validity to larger values of e.

2. First Variational Equations

The first variational equation about the triangular points are (Ref. 9, p. 98)

$$u'' - 2v' = g(e,f)[3u/4 + \omega v] \tag{2.1}$$

$$v'' + 2u' = g(e,f)[\omega u + 9v/4] \tag{2.2}$$

where u and v are the pulsating nondimensional coordinates of the third body relative to one of the triangular points,

$$\omega = 3(3)^{1/2}(\mu - \tfrac{1}{2})/2 \tag{2.3}$$

$$g(e,f) = (1 + e\cos f)^{-1} \tag{2.4}$$

Here f is the true anomaly of the smaller body, and primes denote differentiation with respect to f. Following Szebehely (Ref. 9, p. 254), we introduce a rotation of coordinates according to

$$u = x\cos\beta - y\sin\beta \tag{2.5}$$

$$v = x\sin\beta + y\cos\beta \tag{2.6}$$

where

$$\tan 2\beta = (3)^{1/2}(1 - 2\mu) \tag{2.7}$$

Equations (2.1) and (2.2) become

$$x'' - 2y' - gh_2 x = 0 \tag{2.8}$$

$$y'' + 2x' - gh_1 y = 0 \tag{2.9}$$

where

$$h_{1,2} = \tfrac{3}{2}\{1 \pm [1 - 3\mu(1 - \mu)]^{1/2}\} \tag{2.10}$$

3. Method of Solution

It is known[9] that in the circular case ($e = 0$), infinitesimal motions about the triangular points are stable for $0 \leq \mu < \mu_a = 0.03852$ and unstable for $\mu \geq \mu_a$. Hence, μ_a is the intersection of a transitional curve with the μ axis. Also, it is known from Floquet theory[4] that periodic solutions with periods 2π and 4π correspond to transitional curves. In the interval $0 \leq \mu < \mu_a$ the period 2π corresponds to $\mu = 0$, whereas 4π corresponds to $\mu_b = 0.02589$. Therefore, there are transitional curves that intersect the μ axis at $\mu = 0$ and μ_b.

To determine the transitional curves, we expand x, y, and μ in powers of e where the zeroth-order term for μ is $\mu = 0$, μ_a or μ_b. In the case of μ_b, we take the zeroth-order terms to be periodic with period 4π. The value of μ_a corresponds to a double root for the period, and we take the zeroth-order terms to be the nonsecular solutions. Thus, we let

$$x = \sum_{n=0}^{\infty} x_n e^n \tag{3.1}$$

$$y = \sum_{n=0}^{\infty} y_n e^n \tag{3.2}$$

$$\mu = \sum_{n=0}^{\infty} \mu_n e^n \tag{3.3}$$

where μ_0 stands for the value of μ under consideration 0, μ_a, or μ_b. Substituting (3.3) in (2.10) and expanding in powers of e lead to

$$h_1 = \sum_{n=0}^{\infty} a_n(\mu_0, \mu_1, \ldots, \mu_n)e^n \tag{3.4}$$

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$$h_2 = \sum_{n=0}^{\infty} b_n(\mu_0,\mu_1, \ldots, \mu_n)e^n \tag{3.5}$$

where $b_n = -a_n$ for $n \geq 1$.

Substituting (3.1–3.5) in (2.8) and (2.9), and equating coefficients of equal powers of e to zero lead to

$$x''_n - 2y'_n = \sum_{t=0}\sum_{s=0}\sum_{r=0}^{n=r+s+t} x_r b_s \cos^t f \tag{3.6}$$

$$y''_n + 2x'_n = \sum_{t=0}\sum_{s=0}\sum_{r=0}^{n=r+s+t} y_r a_s \cos^t f \tag{3.7}$$

Carrying out the analysis at $\mu = 0$ leads to the result that the e axis is a transitional curve; it corresponds to the two-body problem. On this curve, $e < 1$ correspond to stable solutions whereas $e \geq 1$ correspond to unstable solutions.

4. Transition Curve at $\mu = \mu_b$

For $\mu_0 = \mu_b = 0.02859$, the zeroth-order equations admit the following periodic solutions of period 4π

$$x_0 = \cos\tau, \qquad y_0 = -\alpha \sin\tau \tag{4.1}$$

$$x_0 = \sin\tau, \qquad y_0 = \alpha \cos\tau \tag{4.2}$$

where

$$\tau = f/2, \qquad \alpha = b_0 + \tfrac{1}{4} = 0.3138 \tag{4.3}$$

There are two branches originating at $\mu = \mu_b$ corresponding to the preceding two independent solutions.

Using (4.1), we find that

$$x''_1 - 2y'_1 - b_0x_1 = (b_1 - b_0/2)\cos\tau - \tfrac{1}{2}b_0 \cos 3\tau \tag{4.4}$$

$$y''_1 + 2x'_1 - a_0y_1 = -\alpha(a_1 + a_0/2)\sin\tau + \tfrac{1}{2}\alpha a_0 \sin 3\tau \tag{4.5}$$

In order that (3.1–3.3) be valid for all f, x_n/x_0 and y_n/y_0 must be bounded for all f. The particular solutions for x_1 and y_1 contain secular terms which make x_1/x_0 and y_1/y_0 be unbounded as $f \to \infty$ unless

$$b_1 - b_0/2 = -\alpha^2(a_1 + a_0/2) \tag{4.6}$$

Solving (4.6) leads to

$$b_1 = (b_0 - a_0\alpha^2)/2(1 - \alpha^2) = -0.1250 \tag{4.7}$$

thence,

$$\mu_1 = -0.05641 \tag{4.8}$$

The solutions for x_1 and y_1, neglecting the homogeneous solutions, are

$$x_1 = 0.5159 \cos 3\tau \tag{4.9}$$

$$y_1 = 0.1569 \sin\tau - 0.3873 \sin 3\tau \tag{4.10}$$

Substituting the zeroth- and first-order solutions into the second-order equation gives

$$x''_2 - 2y'_2 - b_0x_2 = (b_2 + 0.07795)\cos\tau + 0.01397 \cos 3\tau - 0.00051 \cos 5\tau \tag{4.11}$$

$$y''_2 + 2x'_2 - a_0y_2 = (-\alpha a_2 + 0.3382)\sin\tau - 0.02879 \sin 3\tau + 0.3382 \sin 5\tau \tag{4.12}$$

To eliminate secular terms, we require that

$$b_2 + 0.07795 = \alpha(-\alpha a_2 + 0.3382) \tag{4.13}$$

thence,

$$b_2 = 0.03127, \text{ and } \mu_2 = 0.01504 \tag{4.14}$$

The solutions for x_2 and y_2 become

$$x_2 = -0.05294 \cos 3\tau + 0.05139 \cos 5\tau \tag{4.15}$$

$$y_2 = -0.1092 \sin\tau + 0.03617 \sin 3\tau - 0.06479 \sin 5\tau \tag{4.16}$$

The values of μ_1 and μ_2 obtained here agree with those of Ref. 1.

The third-order equations become

$$x''_3 - 2y'_3 - b_0x_3 = (b_3 - 0.05991)\cos\tau - 0.03325 \cos 3\tau - 0.01172 \cos 5\tau - 0.00139 \cos 7\tau \tag{4.17}$$

$$y''_3 + 2x'_3 - a_0y_3 = (-\alpha a_3 - 0.04361)\sin\tau - 0.07102 \sin 3\tau - 0.04681 \sin 5\tau - 0.07399 \sin 7\tau \tag{4.18}$$

Secular terms are eliminated if

$$b_3 - 0.05991 = \alpha(-\alpha a_3 - 0.04361) \tag{41.9}$$

Therefore,

$$b_3 = 0.05127, \ \mu_3 = 0.02257 \tag{4.20}$$

The solutions for x_3 and y_3 are

$$x_3 = -0.01355 \cos 3\tau - 0.00383 \cos 5\tau - 0.00360 \cos 7\tau \tag{4.21}$$

$$y_3 = 0.00864 \sin\tau + 0.02153 \sin 3\tau + 0.00718 \sin 5\tau + 0.00653 \sin 7\tau \tag{4.22}$$

The equations for x_4 and y_4 become

$$x''_4 - 2y'_4 - b_0x_4 = (b_4 + 0.02137)\cos\tau + \text{higher harmonics} \tag{4.23}$$

$$y''_4 + 2x'_4 - a_0y_4 = (-\alpha a_4 - 0.00073)\sin\tau + \text{higher harmonics} \tag{4.24}$$

Elimination of the secular terms yields

$$(b_4 + 0.02137) = \alpha(-\alpha a_4 - 0.00073) \tag{4.25}$$

Hence,

$$b_4 = -0.02396, \ \mu_4 = -0.01231 \tag{4.26}$$

The equation of one of the branches originating at μ_b is

$$\mu^{(1)} = 0.02859 - 0.05641e + 0.01504e^2 + 0.02257e^3 - 0.01278e^4 + 0(e^5) \tag{4.27}$$

Using the periodic solution (4.2) as the zeroth-order term rather than (4.1) leads to the second branch

$$\mu^{(2)} = \mu^{(1)}(-e) \tag{4.28}$$

Equations (4.27) and (4.28) show that if any of the branches were reflected in the μ axis, then the reflection would form a completely smooth continuation of the other branch.

5. Transition curve at $\mu = \mu_a$

When $\mu = \mu_a = 0.03852$, there are two secular and two non-secular independent solutions for x_0 and y_0. Using any of these independent solutions as the zeroth-order term in the expansion leads to the same transition curve because there is only one branch originating at μ_a. Thus, we let

$$x_0 = \cos\sigma f, \qquad y_0 = -\alpha \sin\sigma f \tag{5.1}$$

$$\alpha = (\alpha^2 + b_0)/2\sigma, \qquad \sigma = 1(2)^{1/2} \tag{5.2}$$

Substituting the zeroth-order term into the first-order equations gives

$$x''_1 - 2y'_1 - b_0x_1 = b_1 \cos\sigma f - \frac{1}{2} b_0 \sum_{i=1}^{2} \cos\lambda_i f \tag{5.3}$$

$$y''_1 + 2x'_1 - a_0 y_1 = -\alpha a_1 \sin\sigma f + \frac{1}{2}\alpha a_0 \sum_{i=1}^{2} \sin\lambda_i f \quad (5.4)$$

$$\lambda_{1,2} = \sigma \pm 1 \quad (5.5)$$

In order that x_1/x_0 and y_1/y_0 be bounded for all f, we require that

$$b_1 = 0 \text{ and hence } \mu_1 = 0 \quad (5.6)$$

The solutions for x_1 and y_1 become

$$x_1 = \sum_{i=1}^{2} R_i \cos\lambda_i f \quad (5.7)$$

$$y_1 = \sum_{i=1}^{2} S_i \sin\lambda_i f \quad (5.8)$$

where

$$R_i = [(\lambda_i^2 + a_0)b_0 + 2\alpha a_0 \lambda_i]/2D_i \quad (5.9)$$

$$S_i = -[2b_0\lambda_i + \alpha a_0(\lambda_i^2 + b_0)]/2D_i \quad (5.10)$$

$$D_i = (\lambda_i^2 - \tfrac{1}{2})^2 \quad (5.11)$$

The substitution of the zeroth- and first-order solutions into the second-order equations yields

$$x''_2 - 2y'_2 - b_0 x_2 = P\cos\sigma f + \text{nonsecular producing terms} \quad (5.12)$$

$$y''_2 + 2x'_2 - a_0 y_2 = Q\sin\sigma f + \text{nonsecular producing terms} \quad (5.13)$$

where

$$P = b_2 - b_0(R_1 + R_2 - 1)/2 \quad (5.14)$$

$$Q = \alpha b_2 - a_0(S_1 + S_2 + \alpha)/2 \quad (5.15)$$

The solutions for x_2 and y_2 will contain secular terms unless

$$P = \alpha Q \quad (5.16)$$

Therefore,

$$b_2 = (2)^{1/2}/8,\ \mu_2 = 0.08025 \quad (5.17)$$

The equation for the transition curve originating at μ_a is

$$\mu = 0.03852 - 0.08025e^2 + 0(e^3) \quad (5.18)$$

This expansion agrees with that obtained in Ref. 1 using the method of multiple scales, and reproduces the numerical solution of Ref. 3. Since this second-order expansion reproduces the numerical solution, there is no need to carry out the expansion to higher orders.

6. Concluding Remarks

A comparison between the transition curves obtained here and those obtained numerically by Danby is shown in Fig. 1. The analytical curve originating at $\mu = \mu_a = 0.03852$, although of $0(e^2)$, is indistinguishable from the corresponding numerical curve. At $\mu = \mu_b = 0.02859$, the branch $\mu^{(2)}$ [it is $0(e^4)$] is also indistinguishable from the corresponding numerical branch. The second branch reproduces the numerical curve up to $e = 0.50$. Beyond this value, the two curves deviate from each other; at $\mu = 0$, the analytical value for e is 0.75 compared to the numerically obtained value of 1.0.

In order to improve the convergence of the series expansion for $\mu^{(1)}$, we applied Shanks' nonlinear transformation to recast this series into rational fractions. Shanks found that these rational fractions are often more accurate than the original series. From the series for $\mu^{(1)}$, we may form rational fractions in various ways. Recasting this series into a cubic divided by a linear term yields

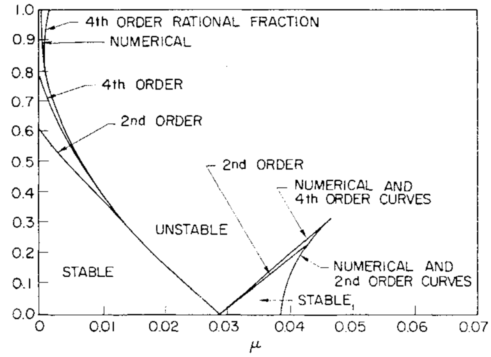

Fig. 1 Comparison of the analytical and numerical transition curves for infinitesimal motions about the triangular points in the elliptic restricted problem of three bodies.

$$\mu = 0.02859(1 - 1.428e - 0.550e^2 + 1.076e^3)/(1 - 0.545e) \quad (6.1)$$

Equation (6.1) is indistinguishable from the numerical curve up to $e = 0.80$ as can be seen from Fig. 1, and, hence, it improves the convergence of the series. However, recasting this series as the ratio of two quadratic expressions yields

$$\mu = \frac{0.02859 - 0.04909e + 0.01396e^2}{1 + 0.2527e + 0.4684e^2} \quad (6.2)$$

which is less accurate than the original series.

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